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Analysis of nonlinear vibrations of a two-degree-of-freedom mechanical system with damping modelled by a fractional derivative

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Abstract. Free damped vibrations of a mechanical two-degree-of-freedom system are considered under the conditions of one-to-one or two-to-one internal resonance, *i.e.*, when natural frequencies of two modes – a mode of vertical vibrations and a mode of pendulum vibrations – are approximately equal to each other or when one natural frequency is nearly twice as large as another natural frequency. Damping features of the system are defined by the fractional derivatives with fractional parameters (the orders of the fractional derivatives) changing from zero to one. It is assumed that the amplitudes of vibrations are small but finite values, and the method of multiple scales is used as a method of solution. The model put forward allows one to obtain the damping coefficient dependent on the natural frequency of vibrations, so it has been shown that the amplitudes of vertical and pendulum vibrations attenuate by an exponential law with damping ratios which are exponential functions of the natural frequencies. Damped soliton-like solutions have been found analytically.

Key words: nonlinear vibrations, fractional derivative, 2dof mechanical system, energy exchange.

1. Introduction

It is known that the transfer of energy from one type to another is made possible during vibrational processes in nonlinear systems. This is called *energy exchange* [1–2]. This phenomenon is particularly evident in modern engineering structures which are very light and flexible due to application of present-day materials, resulting in finite displacements of individual structural elements as well as of the structure as a whole. Among such constructions are suspension-combined systems: suspension and cable-stayed bridges, suspension roofs, etc.

Investigations on energy exchange originate from the paper by Vitt and Gorelik [3], wherein the authors studied small nonlinear vibrations of a two-degree-of-freedom (2dof) system consisting of a load suspended on a linearly elastic spring and executing pendulum vibrations and vibrations along the spring's axis in the same vertical plane. In spite of the apparent simplicity of that system, it realistically explains some phenomena occurring during vibrations of more complex nonlinear systems, and in particular, describes all types of energy exchange from pendulum vibratory motions into oscillatory motions along the spring's axis, and *vice versa*: the periodic and aperiodic energy interchange, as well as stationary regimes during which energy exchange is absent.

The energy exchange mechanism in a similar nonlinear 2dof system has been studied by Sado [4] and Shitikova [5]. The system was made up of two loads, one of which was suspended on a linearly elastic spring and executed vertical vibrations; the other was suspended on an unstretched rod and executed pendulum vibrations in the same vertical plane. Reviews devoted

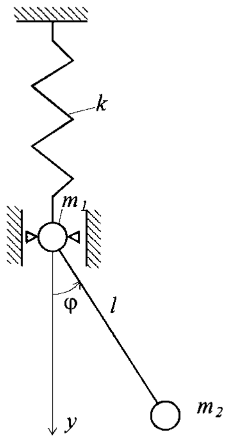

Figure 1. Scheme of a 2dof mechanical system.

to nonlinear vibrations of 2dof systems can be found in Nayfeh and Balachandran [1] and Sado [2].

The experimental data obtained by Abdel-Ghaffar and Housner [6] and Abdel-Ghaffar and Scanlan [7] during ambient vibration studies of the Vincent-Thomas Suspension Bridge and the Golden Gate Bridge, respectively, show that different vibrational modes feature different amplitude damping factors, and the order of smallness of these coefficients tells about low damping capacity of suspension-combined systems, resulting in prolonged energy transfer from one partial subsystem to another. Besides, as the natural vibration frequencies increase, the corresponding damping ratios decrease.

Introducing common linear viscosity (when damping features of the system are prescribed by the first derivative of the displacement with respect to time) into the nonlinear 2dof system results in damping of the energy exchange process [8–9]; however, in this case the damping coefficient is independent of the frequency of vibrations, but this result is in conflict with the experimental data presented in [6–7].

To get the theoretical investigations in line with the experiment, in the given paper, fractional derivatives are introduced for describing the processes of internal friction proceeding in a 2dof mechanical system at free vibrations. The model put forward allows one to obtain the damping coefficient dependent on the natural frequency of vibrations. The model proposed describes realistically the dynamic behaviour of many nonlinear systems with more than one degree-of-freedom, among them multi-degree-of-freedom suspension-combined systems in the special case that only two interacting modes dominate in the vibratory motion.

2. Problem formulation

Consider nonlinear free damped vibrations of a two-degree-of-freedom mechanical system presented in Figure 1. Assume that such a system vibrates in a viscous medium whose viscous features are defined by fractional derivatives. In this case the damping forces acting on the mass m_1 and the pendulum of length l and mass m_2 are, respectively, the following

$$Q_1 = \beta \mathrm{D}^{\gamma_1} y, \qquad Q_2 = \beta l \mathrm{D}^{\gamma_2} \varphi, \tag{1}$$

where y is the displacement of the mass m_1, φ is the deflection angle of the pendulum, β is the viscous coefficient of the medium, the damping force Q_1 is directed vertically, the damping force Q_2 is directed tangentially to the trajectory of the mass m_2, the fractional derivative $D^\gamma x$ ($x = y$ or φ) is defined as follows [10, pp. 41–44]

$$D^\gamma x = \frac{d}{dt}\int_0^t \frac{x(u)du}{\Gamma(1-\gamma)(t-u)^\gamma}, \quad (0 < \gamma \leqslant 1), \tag{2}$$

where γ is the order of the fractional derivative (fractional parameter), and $\Gamma(1-\gamma)$ is the Gamma-function.

Then the equations of motion of such a system in dimensionless form, accurate to within values of second-order smallness, can be written as

$$\ddot{y}^* + \beta D^{\gamma_1} y^* + \omega_0^{*2} y^* - a\dot{\varphi}^2 - a\varphi\ddot{\varphi} = 0, \tag{3a}$$

$$\ddot{\varphi} + \beta D^{\gamma_2}\varphi + \Omega_0^{*2}\varphi - b\varphi\ddot{y}^* = 0, \tag{3b}$$

where an overdot denotes a differentiation with respect to the time t,

$$\omega_0^{*2} = \omega_0^2\frac{y_0}{g}, \qquad \Omega_0^{*2} = \Omega_0^2\frac{y_0}{g}, \qquad \omega_0 = \sqrt{\frac{k}{m_1+m_2}}, \qquad \Omega_0 = \sqrt{\frac{g}{l}}, \qquad y_0 = \frac{m_1 g}{k},$$

$$y^* = \frac{y}{y_0}, \qquad t^* = t\sqrt{\frac{g}{y_0}}, \qquad a = \frac{m_2}{m_1+m_2}\frac{l}{y_0} = \frac{m_2}{m_1}\frac{\omega_0^2}{\Omega_0^2}, \qquad b = \frac{y_0}{l} = \frac{m_1}{m_1+m_2}\frac{\Omega_0^2}{\omega_0^2},$$

where g is acceleration of gravity, and k is the spring rigidity. For ease of presentation, asterisks will be omitted henceforth.

Assume that the linear natural frequency ω_0 and the linear natural frequency Ω_0 are equal or approximately equal to each other (the case of the one-to-one internal resonance)

$$\omega_0 = \Omega_0 + \varepsilon^2\sigma \tag{4a}$$

or the linear natural frequency ω_0 is nearly twice as large as the linear natural frequency Ω_0 (the case of the two-to-one internal resonance), such that

$$\omega_0 = 2\Omega_0 + \varepsilon\sigma, \tag{4b}$$

where σ is a detuning parameter, and ε is a small parameter which is of the same order of magnitude as the amplitudes.

The set of Equations (3) with due account for relationship (4a) or (4b) describes the two processes which are related to each other and go on concurrently: the energy-exchange mechanism between vertical vibrations of the mass m_1 and the pendulum's vibrations, and the process of energy dissipation during this interaction. Since further investigations will be carried out by the method of multiple scales and these two processes should proceed on the same time scale, then it is necessary to assume that the viscosity coefficient β has the order of ε^2 in the case of the one-to-one internal resonance or the order of ε in the case of the two-to-one internal resonance, *i.e.*, it can be represented as $\beta = \varepsilon^2\mu$ or $\beta = \varepsilon\mu$, respectively, where μ is a finite value. At other orders of smallness of the viscosity coefficient, energy dissipation will occur either too fast (when the order of smallness is larger than ε^2 in the case of the one-to-one internal resonance or the order of ε in the case of the two-to-one internal resonance) or too slow (when the order of smallness is less than ε^2 in the case of the one-to-one internal resonance or the order of ε in the case of the two-to-one internal resonance) relative to the process of energy exchange.

A fractional derivative is the immediate extension of an ordinary derivative. In fact, when $\gamma \to 1$, the function $[\Gamma(1-\gamma)(t-u)^{\gamma}]^{-1}$ tends to the Dirac δ-function $\delta(t-u)$, and hence $D^{\gamma}x$ tends to $\dot{x}$, *i.e.*, at $\gamma \to 1$ the fractional derivative goes over into the ordinary derivative, and the mathematical model of the 2dof mechanical system under consideration transforms into the Kelvin–Voigt model, wherein the elastic element behaves nonlinearly but the viscous element behaves linearly. When $\gamma \to 0$, the fractional derivative $D^{\gamma}x$ tends to $x(t)$. To put it otherwise, the introduction of the new fractional parameter γ along with the parameter μ allows one to change not only the magnitude of viscosity at the cost of an increase or decrease in the parameter μ, but also the character of viscosity at the sacrifice of variations in the fractional parameter.

3. Method of solution

An approximate solution of Equations (3) for small amplitudes weakly varying with time can be represented by an expansion in terms of different time scales in the following form [11, Chapter 6]

$$y(t) = \varepsilon y_1(T_0, T_1, T_2, \ldots) + \varepsilon^2 y_2(T_0, T_1, T_2, \ldots) + \varepsilon^3 y_3(T_0, T_1, T_2, \ldots) + \cdots, \tag{5a}$$

$$\varphi(t) = \varepsilon \varphi_1(T_0, T_1, T_2, \ldots) + \varepsilon^2 \varphi_2(T_0, T_1, T_2, \ldots) + \varepsilon^3 \varphi_3(T_0, T_1, T_2, \ldots) + \cdots, \tag{5b}$$

where $T_n = \varepsilon^n t$ $(n = 0, 1, 2, \ldots)$ are new independent variables.

Considering that

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \qquad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \cdots$$

and substituting (5) in (3), after equating the coefficients at like powers of ε to zero, we obtain to order ε

$$D_0^2 y_1 + \omega_0^2 y_1 = 0, \qquad D_0^2 \varphi_1 + \Omega_0^2 \varphi_1 = 0; \tag{6a,b}$$

to order ε^2:

$$D_0^2 y_2 + \omega_0^2 y_2 = -2D_0 D_1 y_1 + a\varphi_1 D_0^2 \varphi_1 + a(D_0 \varphi_1)^2 - \mu(2-k) D_0^{\gamma_1} y_1, \tag{7a}$$

$$D_0^2 \varphi_2 + \Omega_0^2 \varphi_2 = -2D_0 D_1 \varphi_1 + b\varphi_1 D_0^2 y_1 - \mu(2-k) D_0^{\gamma_2} \varphi_1; \tag{7b}$$

to order ε^3:

$$\begin{aligned} D_0^2 y_3 + \omega_0^2 \varphi_3 \;=\; & -2D_0 D_1 y_2 - (D_1^2 + 2D_0 D_2) y_1 + a\varphi_1 (D_0^2 \varphi_2 + 2D_0 D_1 \varphi_1) \\ & + a\varphi_2 D_0^2 \varphi_1 + 2a D_0 \varphi_1 (D_1 \varphi_1 + D_0 \varphi_2) - \mu(k-1) D_0^{\gamma_1} y_1, \end{aligned} \tag{8a}$$

$$\mathrm{D}_0^2\varphi_3 + \Omega_0^2\varphi_3 = -2\mathrm{D}_0\mathrm{D}_1\varphi_2 - (\mathrm{D}_1^2 + 2\mathrm{D}_0\mathrm{D}_2)\varphi_1 + b\varphi_1(\mathrm{D}_0^2\varphi_2 + 2\mathrm{D}_0\mathrm{D}_1 y_1) + b\varphi_2\mathrm{D}_0^2 y_1 - \mu(k-1)\mathrm{D}_0^{\gamma_2}\varphi_1, \tag{8b}$$

where $\mathrm{D}_n = \partial/\partial T_n$.

The sets of Equations (6)–(8) describe the cases of the one-to-one and two-to-one internal resonances at $k = 2$ and $k = 1$, respectively.

During the deduction of Equations (6)–(8) it was assumed that

$$\begin{aligned}\mathrm{D}^\gamma &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\gamma = (\mathrm{D}_0 + \varepsilon\mathrm{D}_1 + \varepsilon^2\mathrm{D}_2 + \cdots)^\gamma \\ &= \mathrm{D}_0^\gamma + \varepsilon\gamma\mathrm{D}_0^{\gamma-1}\mathrm{D}_1 + \varepsilon^2\gamma(\gamma-1)\mathrm{D}_0^{\gamma-2}\mathrm{D}_2 + \cdots,\end{aligned} \tag{9}$$

where D_0^γ is obtained from (2) by replacing t with T_0. Thus, the fractional derivative is interpreted as the fractional power of the differentiation operator. Such a notion of the fractional derivative is used in scientific literature under a proper interpretation for expanding fractional differentiation to some functional spaces (see, for example, Samko *et al.* [10, Chapter 2, Section 5]).

We shall seek the solution of Equations (6) in the form

$$y_1 = A_1(T_1, T_2)\exp(\mathrm{i}\omega_0 T_0) + \bar{A}_1(T_1, T_2)\exp(-\mathrm{i}\omega_0 T_0), \tag{10a}$$

$$\varphi_1 = A_2(T_1, T_2)\exp(\mathrm{i}\Omega_0 T_0) + \bar{A}_2(T_1, T_2)\exp(-\mathrm{i}\Omega_0 T_0), \tag{10b}$$

where A_1 and A_2 are unknown complex functions, and $\bar{A}_1$ and $\bar{A}_2$ are the complex conjugates of A_1 and A_2, respectively.

The influence of the detuning parameter σ on undamped free vibrations was investigated in Rossikhin and Shitikova [12], and hence, in the present paper, we shall examine the influence of damping modelled by a fractional derivative on free damped vibrations of the system under consideration, *i.e.*, we shall put $\sigma = 0$ in further discussion.

3.1. The case of a one-to-one internal resonance

To construct the solution in the case of a one-to-one internal resonance, it will suffice to restrict consideration to the terms of the order of ε^3 and to consider the amplitudes A_1 and A_2 as functions of T_1 and T_2.

In order to eliminate secular terms arising in the second approximation, it would suffice to consider the functions A_1 and A_2 dependent on T_2 only. Then, the solution of Equations (7) at $k = 2$ with due account of (10) takes the form

$$y_2 = \tfrac{2}{3}a\frac{\Omega_0^2}{\omega_0^2}A_2^2\exp(2\mathrm{i}\omega_0 T_0) + cc, \tag{11}$$

$$\varphi_2 = b\frac{\omega_0^2}{\Omega_0^2}(\tfrac{1}{3}A_1A_2\exp(2\mathrm{i}\Omega_0 T_0) - A_2\bar{A}_1) + cc, \tag{12}$$

where cc is the complex conjugate part to the preceding terms.

If we substitute expressions (10)–(12) in the right-hand sides of Equations (8) at $k = 2$, then we are led to a system of equations for determining y_3 and φ_3. The solution should not contain the secular terms, and therefore, the following relationships are to be satisfied

$$-\mathrm{i}\mathrm{D}_2 A_1 - \tfrac{1}{2}\mu\omega_0^{-1}(\mathrm{i}\omega_0)^{\gamma_1} A_1 + ab\omega_0(\tfrac{1}{3}A_1 A_2 \bar{A}_2 + \tfrac{1}{2}\bar{A}_1 A_2^2) = 0, \tag{13}$$

$$-\mathrm{i}\mathrm{D}_2 A_2 - \tfrac{1}{2}\mu\Omega_0^{-1}(\mathrm{i}\Omega_0)^{\gamma_2} A_2 - \tfrac{4}{3}ab\Omega_0 A_2^2\bar{A}_2 + b^2\frac{\omega_0^4}{\Omega_0^3}(\tfrac{1}{3}A_1\bar{A}_1 A_2 + \tfrac{1}{2}A_1^2\bar{A}_2) = 0. \tag{14}$$

When deducing Equations (13) and (14), we used the known formula from fractional calculus [10, Table 9.1, p. 140]

$$\mathrm{D}_0^{\gamma}\, \mathrm{e}^{aT_0} = a^{\gamma}\, \mathrm{e}^{aT_0} + \frac{\sin\pi\gamma}{\pi}\int_0^{\infty}\frac{u^{\gamma}\,\mathrm{e}^{-uT_0}\,\mathrm{d}u}{u+a}. \tag{15}$$

The second term in formula (15) can be omitted in comparison with the first one in the two cases: when the value γ is small, for example of the order of ε, or when the transient processes, which define only the drift of the equilibrium position around which the vibration motion occurs [13], are neglected. The first assumption is valid for suspension bridges [14] and is verified by experimental data [6–7]. In the present paper, we consider the second case.

Let us multiply Equations (13) and (14) by $\bar{A}_1$ and $\bar{A}_2$, respectively, and find their complex conjugates. Adding every pair of the mutually adjoint equations with each other and subtracting one from the other, we obtain as a result

$$\mathrm{i}(A_1\mathrm{D}_2\bar{A}_1 - \bar{A}_1\mathrm{D}_2A_1) - \mu\omega_0^{\gamma_1-1}A_1\bar{A}_1\cos(\tfrac{1}{2}\pi\gamma_1)$$
$$+ab\omega_0(\tfrac{2}{3}A_1\bar{A}_1A_2\bar{A}_2 + \tfrac{1}{2}(\bar{A}_1^2A_2^2 + A_1^2\bar{A}_2^2)) = 0, \tag{16a}$$

$$\mathrm{i}(A_2\mathrm{D}_2\bar{A}_2 - \bar{A}_2\mathrm{D}_2A_2) - \mu\Omega_0^{\gamma_2-1}A_2\bar{A}_2\cos(\tfrac{1}{2}\pi\gamma_2)$$
$$+b\omega_0(\tfrac{2}{3}aA_1\bar{A}_1A_2\bar{A}_2 - \tfrac{8}{3}aA_2^2\bar{A}_2^2 + \tfrac{1}{2}b(A_1^2\bar{A}_2^2 + \bar{A}_1^2A_2^2)) = 0, \tag{16b}$$

$$\mathrm{i}(A_1\mathrm{D}_2\bar{A}_1 + \bar{A}_1\mathrm{D}_2A_1) + \mathrm{i}\mu\omega_0^{\gamma_1-1}A_1\bar{A}_1\sin(\tfrac{1}{2}\pi\gamma_1) + \tfrac{1}{2}ab\omega_0(A_1^2\bar{A}_2^2 - \bar{A}_1^2A_2^2) = 0, \tag{16c}$$

$$\mathrm{i}(A_2\mathrm{D}_2\bar{A}_2 + \bar{A}_2\mathrm{D}_2A_2) + \mathrm{i}\mu\Omega_0^{\gamma_2-1}A_2\bar{A}_2\sin(\tfrac{1}{2}\pi\gamma_2) + \tfrac{1}{2}b^2\omega_0(\bar{A}_1^2A_2^2 - A_1^2\bar{A}_2^2) = 0. \tag{16d}$$

Representing the functions A_1 and A_2 in polar form

$$A_1 = a_1\exp(\mathrm{i}\varphi_1), \qquad A_2 = a_2\exp(\mathrm{i}\varphi_2), \tag{17}$$

we rewrite Equations (16) as

$$(a_1^2)^{\cdot} + s_1a_1^2 - ab\omega_0a_1^2a_2^2\sin\delta = 0, \tag{18a}$$

$$(a_2^2)^{\cdot} + s_2a_2^2 + b^2\omega_0a_1^2a_2^2\sin\delta = 0, \tag{18b}$$

$$\dot{\varphi}_1 - \tfrac{1}{2}\sigma_1 + ab\omega_0a_2^2(\tfrac{1}{3} + \tfrac{1}{2}\cos\delta) = 0, \tag{18c}$$

$$\dot{\varphi}_2 - \tfrac{1}{2}\sigma_2 + ab\omega_0(\tfrac{1}{3}a_1^2 - \tfrac{4}{3}a_2^2 + \frac{1}{2}\,\frac{b}{a}a_1^2\cos\delta) = 0, \tag{18d}$$

where $(a_i^2)^{\cdot} = 2a_i\dot{a}_i (i = 1, 2)$, an overdot denotes differentiation with respect to T_2, $\delta = 2(\varphi_2 - \varphi_1)$, and

$$s_1 = \mu\omega_0^{\gamma_1 - 1}\sin(\tfrac{1}{2}\pi\gamma_1), \qquad s_2 = \mu\Omega_0^{\gamma_2 - 1}\sin(\tfrac{1}{2}\pi\gamma_2),$$

$$\sigma_1 = \mu\omega_0^{\gamma_1 - 1}\cos(\tfrac{1}{2}\pi\gamma_1), \qquad \sigma_2 = \mu\Omega_0^{\gamma_2 - 1}\cos(\tfrac{1}{2}\pi\gamma_2).$$

The obtained set of Equations (18) differs from the similar set presented in [9] for an ordinary linear damping in that it describes the vibratory motions with the damping coefficient depending on the vibration frequency. This phenomena is verified by experimental data obtained for nonlinear systems [6–7, 14].

Equations (18a) and (18b) can be rewritten as

$$\frac{\left(a_1^2\right)^{\cdot} + s_1 a_1^2}{\left(a_2^2\right)^{\cdot} + s_2 a_2^2} = -\frac{a}{b}. \tag{19}$$

Introducing new functions $\xi(T_2)$ and $\eta(T_2)$, such that

$$a_1^2 = E_0\xi(T_2)\exp(-s_1T_2), \qquad a_2^2 = E_0\eta(T_2)\exp(-s_2T_2), \tag{20}$$

where E_0 is the initial system's energy, and substituting (20) in (19), we have

$$\frac{\dot{\xi}\exp(-s_1T_2)}{\dot{\eta}\exp(-s_2T_2)} = -\frac{a}{b}. \tag{21}$$

The solution to (21) gives the following relation between the functions $\xi(T_2)$ and $\eta(T_2)$

$$\eta(T_2) = \frac{b}{a}(1-\xi)\,\mathrm{e}^{(s_2 - s_1)T_2} - \frac{b}{a}(s_2 - s_1)\int_0^{T_2}(1-\xi)\,\mathrm{e}^{(s_2-s_1)T_2}\,\mathrm{d}T_2. \tag{22}$$

Then, Equations (18) take the form

$$\dot{\xi} - b^2\omega_0E_0\xi\left[(1-\xi)\,\mathrm{e}^{-s_1T_2} - (s_2 - s_1)\,\mathrm{e}^{-s_2T_2}\int_0^{T_2}(1-\xi)\,\mathrm{e}^{(s_2-s_1)T_2}\,\mathrm{d}T_2\right]\sin\delta = 0, \tag{23a}$$

$$\dot{\delta} - (\sigma_2 - \sigma_1) + \tfrac{2}{3}ab\omega_0(a_1^2 - 5a_2^2) + ab\omega_0\left(\frac{b}{a}a_1^2 - a_2^2\right)\cos\delta = 0. \tag{23b}$$

The initial conditions

$$\xi|_{T_2=0} = \xi_0, \qquad \delta|_{T_2=0} = \delta_0 \tag{24}$$

should be added to (23).

The nonlinear set of Equations (23) with the initial conditions (24) completely describe the vibrational process of the mechanical system being investigated under the condition of the internal resonance one-to-one. The set of Equations (23) with due account for (20), (22) and (24) can be solved numerically.

In the partial case when $\gamma_1 = \gamma_2 = \gamma$, the solution can be reduced to the calculation of an incomplete elliptic integral of the first kind. Thus, multiplying (18a) and (18b) by b and a, respectively, then adding them, and using (4a), we obtain

$$\dot{E} + \mu\omega_0^{\gamma-1} E \sin(\tfrac{1}{2}\pi\gamma) = 0, \tag{25a}$$

$$E = a_1^2 + ab^{-1}a_2^2, \tag{25b}$$

where E is the energy of the system.

Integration of (25) yields

$$E = E_0 \exp(-sT_2), \qquad s = s_1|_{\gamma_1=\gamma}. \tag{26a,b}$$

Formulas (26) show that, owing to the fractional parameter γ, dissipation of the system's energy depends on the natural frequency of vibrations.

When $\gamma \to 1$, the damping value s tends to the viscosity coefficient μ, and from (26) it follows that

$$E = E_0 \exp(-\mu T_2), \qquad s = \mu. \tag{27a,b}$$

Reference to (27) shows that in the case $\gamma = 1$ the damping coefficient is independent of the natural frequency ω_0.

When $\gamma_1 = \gamma_2 = \gamma$, Equations (20) and (23) take, respectively, the form

$$a_1 = \sqrt{E_0\xi}\,\exp(-\tfrac{1}{2}sT_2), \qquad a_2 = \sqrt{E_0(1-\xi)ba^{-1}}\,\exp(-\tfrac{1}{2}sT_2), \tag{28a,b}$$

$$\dot{\xi} = \omega_0 E_0 b^2 \xi(1-\xi)\exp(-sT_2)\ \sin\delta, \tag{29a}$$

$$\dot{\delta} = b^2\omega_0 E_0 \left[(1-2\xi)\cos\delta - \frac{2}{3}\,\frac{a}{b}\,\xi + \tfrac{10}{3}\,(1-\xi)\right]\exp(-sT_2). \tag{29b}$$

Dividing (29b) by (29a), we arrive at

$$\frac{\mathrm{d}\cos\delta}{\mathrm{d}\xi} + \frac{1-2\xi}{\xi(1-\xi)}\cos\delta = \frac{2a}{3b}\,\frac{1}{1-\xi} - \frac{10}{3}\,\frac{1}{\xi}. \tag{30}$$

The solution of (30) has the form

$$G_1(\xi,\delta) = \xi(1-\xi)\cos\delta - \frac{a}{3b}\,\xi^2 - \tfrac{5}{3}\,(1-\xi)^2 = G_1^0, \tag{31}$$

where $G_1^0 = G_1(\xi_0, \delta_0)$ is an arbitrary constant.

Eliminating the value δ from (29a), taking due account of (31), and integrating over T_2, we have

$$\frac{1}{\sqrt{m_1 m_2}} \int_{\xi_0}^{\xi} \frac{\mathrm{d}\xi}{\left[(\xi^2 + p_1\xi + q_1)(\xi^2 + p_2\xi + q_2)\right]^{1/2}} = \frac{\omega_0 b^2 E_0}{s}\left[1 - \exp(-sT_2)\right], \tag{32}$$

where

$$m_1 = -\tfrac{8}{3} - \frac{a}{3b}, \qquad m_2 = \tfrac{2}{3} + \frac{a}{3b}, \qquad p_1 = \frac{13}{3m_1}, \qquad p_2 = -\frac{7}{3m_2},$$

$$q_1 = -(G_1^0 + \tfrac{5}{3})m_1^{-1}, \qquad q_2 = (G_1^0 + \tfrac{5}{3})m_2^{-1}.$$

The integral on the left-hand side of (32) can be transformed into an incomplete elliptic integral of the first kind [15, Chapter 17].

Let us investigate the particular solution of (29) corresponding to the case $\dot{\xi} = 0$. Then it follows from (29a) that $\sin\delta = 0$, *i.e.*, $\delta = 0 \pm \pi n(n = 0, 1, 2, \ldots)$. Considering this magnitude of δ in (28b), we are led to the following relationship for ξ

$$\xi = \xi_0^{\pm} = \frac{\pm 1 + \frac{10}{3}}{2(\pm 1 + \frac{a}{3b} + \frac{5}{3})}, \tag{33}$$

which vanishes the expression in square brackets in (29b).

Substituting the found magnitude of $\xi_0^{\pm}$ (33) in (28), we obtain the expressions for the amplitudes of vibrations

$$a_1 = (a_1)_0 \exp\left(\frac{-sT_2}{2}\right), \qquad a_2 = (a_2)_0 \exp\left(\frac{-sT_2}{2}\right), \tag{34}$$

where $(a_1)_0$ and $(a_2)_0$ are the initial magnitudes of the amplitudes a_1 and a_2, respectively. From (34) it is seen that the obtained particular solution describes the aperiodic regime without energy exchange.

Putting $\mu = 0$ in all equations, we can obtain the solution for free undamped vibrations of the system under consideration being under the conditions of the one-to-one internal resonance.

3.2. The case of a two-to-one internal resonance

Now consider the case (4b), *i.e.*, when a two-to-one internal resonance takes place. To construct the solution in the case of the two-to-one internal resonance, it will suffice to restrict consideration to the terms of the order of ε^2 and to consider the amplitudes A_1 and A_2 as functions of T_1 only.

In order to eliminate secular terms arising in the second approximation, it would suffice to consider the following equalities:

$$2\mathrm{i}\omega_0 \mathrm{D}_1 A_1 + \mu A_1 (\mathrm{i}\omega_0)^{\gamma_1} + 2aA_2^2\Omega_0^2 = 0, \tag{35}$$

$$2\mathrm{i}\Omega_0 \mathrm{D}_1 A_2 + \mu A_2 (\mathrm{i}\Omega_0)^{\gamma_2} + bA_1\bar{A}_2\omega_0^2 = 0. \tag{36}$$

As a result of the procedure used for the deduction of (18), we obtain from relations (35) and (36) with due account of (4b)

$$(a_1^2)^{\cdot} + s_1 a_1^2 + \frac{1}{2} a\omega_0 a_1 a_2^2 \sin\delta = 0, \tag{37a}$$

$$(a_2^2)^{\cdot} + s_2 a_2^2 - 2b\omega_0 a_1 a_2^2 \sin\delta = 0, \tag{37b}$$

$$\dot{\varphi}_1 - \tfrac{1}{2}\sigma_1 - \tfrac{1}{4}a\omega_0 a_2^2 a_1^{-1}\cos\delta = 0, \tag{37c}$$

$$\dot{\varphi}_2 - \tfrac{1}{2}\sigma_2 - b\omega_0 a_1\cos\delta = 0, \tag{37d}$$

where $(a_i^2)^{\cdot} = 2a_i\dot{a}_i (i = 1, 2)$, an overdot denotes differentiation with respect to T_1, and $\delta = 2\varphi_2 - \varphi_1$.

The obtained set of (37) differs from the similar set presented in [9] for an ordinary linear damping in that it describes the vibratory motions with the damping coefficient depending on the vibration frequency. This phenomena is verified by experimental data obtained for nonlinear systems [6–7, 14].

Equations (37a) and (37b) can be rewritten as

$$\frac{(a_1^2)^{\cdot} + s_1 a_1^2}{(a_2^2)^{\cdot} + s_2 a_2^2} = -\frac{1}{4}\,\frac{a}{b}. \tag{38}$$

Introducing new functions $\xi(T_1)$ and $\eta(T_1)$, such that

$$a_1^2 = E_0\xi(T_1)\exp(-s_1T_1), \qquad a_2^2 = E_0\eta(T_1)\exp(-s_2T_1), \tag{39}$$

and substituting (39) in (38) yields

$$\frac{\dot{\xi}\exp(-s_1T_2)}{\dot{\eta}\exp(-s_2T_2)} = -\frac{1}{4}\,\frac{a}{b}. \tag{40}$$

The solution to (40) gives the following relation between the functions $\xi(T_1)$ and $\eta(T_1)$

$$\eta(T_1) = 4\frac{b}{a}(1-\xi)\,\mathrm{e}^{(s_2-s_1)T_1} - 4\frac{b}{a}(s_2-s_1)\int_0^{T_1}(1-\xi)\,\mathrm{e}^{(s_2-s_1)T_1}\,\mathrm{d}T_1. \tag{41}$$

Then the set of (37) takes the form

$$\dot{\xi} + 2b\omega_0 E_0^{1/2}\xi^{1/2}\,\mathrm{e}^{s_1T_1/2} \times\left[(1-\xi)\,\mathrm{e}^{-s_1T_1} - (s_2-s_1)\,\mathrm{e}^{-s_2T_1}\int_0^{T_1}(1-\xi)\,\mathrm{e}^{(s_2-s_1)T_1}\,\mathrm{d}T_1\right]\sin\delta = 0, \tag{42a}$$

$$\dot{\delta} - (\sigma_2 - \tfrac{1}{2}\,\sigma_1) + \omega_0 E_0^{1/2}\xi^{-1/2}(\tfrac{1}{4}a\eta\,\mathrm{e}^{(s_1/2-s_2)T_1} - 2b\xi\,\mathrm{e}^{-s_1T_1/2})\cos\delta = 0. \tag{42b}$$

The initial conditions

$$\xi|_{T_1=0} = \xi_0, \qquad \delta|_{T_1=0} = \delta_0 \tag{43}$$

should be added to (42).

The nonlinear set of Equations (42) with the initial conditions (43) completely describe the vibrational process of the mechanical system being investigated under the condition of the internal resonance two-to-one. Equations (42) with due account for (41) and (43) can be solved numerically.

In the particular case when $\mu = 0$, *i.e.*, damping effect is neglected, (42) take the form

$$\dot{\xi} = -B\sqrt{\xi}(1-\xi)\sin\delta, \tag{44a}$$

$$\dot{\delta} = B\left(\sqrt{\xi} - \frac{1-\xi}{2\sqrt{\xi}}\right)\cos\delta, \tag{44b}$$

where $B = 2b\omega_0 E_0^{1/2}$.

With $\dot{\delta} = \dot{\xi}\,\mathrm{d}\delta/\mathrm{d}\xi$ and using (44a) yields

$$\frac{\mathrm{d}\cos\delta}{\mathrm{d}\xi} + \frac{1}{2}\,\frac{1-3\xi}{\xi(1-\xi)} = 0. \tag{45}$$

The solution of (45) has the form

$$\cos\delta = \frac{G_2^0}{\xi^{1/2}(1-\xi)}, \tag{46}$$

where G_2^0 is an arbitrary constant determined from the initial conditions and defined by

$$G_2^0(\xi_0, \delta_0) = \xi_0^{1/2}(1-\xi_0)\cos\delta_0. \tag{47}$$

Eliminating the variable δ from (44a) and (46), and integrating over T_1, we have

$$\int_{\xi_0}^{\xi} \frac{\mathrm{d}\xi}{(\xi^3 - 2\xi^2 + \xi - G_2^{0\,2})^{1/2}} = -BT_1. \tag{48}$$

It may be shown that the integral in (48) can be transformed into an incomplete elliptic integral of the first kind [13, Chapter 17].

If $\xi_0 \to 1$ or $\xi_0 \to 0$, then, as follows from formula (47), $G_2^0 = 0$. Substituting the known value G_2^0 in the integral (48), we obtain

$$\int_{\xi_0}^{\xi} \frac{\mathrm{d}\xi}{(1-\xi)\xi^{1/2}} = -BT_1. \tag{49}$$

It can be seen from (49) with due account for (44) that, when $\xi = \xi_0 = 1$, only phase modulated motions are realized, because $a_1 = \text{const}, a_2 = 0$, but $\delta = -\varphi_1 = -\varphi_{10}$, where φ_{10} is the initial phase of the vibrations. At $\xi_0 \neq 0$, Equation (49) takes the form

$$\log\left|\frac{\xi^{1/2}-1}{\xi^{1/2}+1}\right|\Bigg|_{\xi_0}^{\xi} = -BT_1, \tag{50}$$

or

$$\xi = \left[\frac{1+\xi_0^{1/2} - (1-\xi_0^{1/2})\exp(-BT_1)}{1+\xi_0^{1/2} + (1-\xi_0^{1/2})\exp(-BT_1)}\right]^2,$$

$$\delta(T_1) = \delta_0 = \frac{\pi}{2} + \pi n, \quad n = 0, 1, 2, \ldots\,. \tag{51}$$

Reference to (51) shows that this formula describes pure amplitude-modulated motions, during which the one-sided energy exchange between the vertical and pendulum vibrations takes place such that $\xi \to 1$ as T_1 increases. If $\xi_0 = 0$, then from (51) we obtain the known soliton-like solution in the form of a single kink [16, Chapter 7].

$$\sqrt{\xi} = \tanh(1/2\ BT_1). \tag{52}$$

Physically speaking, this solution-kink is responsible for the one-sided energy exchange when the energy of the pendulum vibration completely transforms into the energy of the vertical vibration with time, so that the pendulum vibrations give way to the vertical vibrations.

From (44a) and (44b) it can be found that $\dot{\xi} = 0$ and $\dot{\delta} = 0$ if

$$\xi_0^{\pm} = \tfrac{1}{3}, \qquad \cos\delta_0^{\pm} = \pm 1, \tag{53}$$

i.e., the initial conditions (53) correspond to the stationary regime when the energy exchange is absent.

4. Numerical results

As an example, we consider vibrations of the system presented in Figure 1 at the following magnitudes of the parameters: in the case of the one-to-one resonance $a = 1{\cdot}7913, a/b = 5, \omega_0^2 = b, \mu = 0{\cdot}005$, and $E_0 = 1$; in the case of the two-to-one resonance $a = 2, a/b = 12, \omega_0^2 = 4b, \mu = 0{\cdot}05$, and $E_0 = 1$.

4.1. Free undamped vibrations

Before we investigate the influence of viscosity on the nonlinear damped vibrations of the system under discussion, we consider the case when $\mu = 0$, i.e., viscosity is absent.

To analyze different vibrational regimes corresponding to different initial conditions, it is convenient to use a hydrodynamic analogy which has been described in detail in [12]. For this purpose we introduce for consideration the phase fluid moving along the plane $\xi\delta$ in the channel of the finite width $(0 \leqslant \xi \leqslant 1)$ and the infinite length $(\infty < \delta < \infty)$ with the velocity $v(v_\xi = \dot{\xi}$ and $v_\delta = \dot{\delta})$. Each point with the coordinates ξ, δ on the phase plane corresponds to certain magnitudes of the amplitudes a_1 and a_2 at the fixed instant, and to the phase difference relative to each other at the same instant. Due to such hydrodynamic analogy, $\dot{\xi}$ and $\dot{\delta}$ are expressed in terms of one and the same stream function in the case of the one-to-one resonance

$$\dot{\xi} = -\omega_0 b^2 E_0 \frac{\partial G_1}{\partial \delta}, \qquad \dot{\delta} = \omega_0 b^2 E_0 \frac{\partial G_1}{\partial \xi} \tag{54}$$

and in the case of the two-to-one resonance

$$\dot{\xi} = B\frac{\partial G_2}{\partial \delta}, \qquad \dot{\delta} = -B\frac{\partial G_2}{\partial \xi}, \tag{55}$$

where G_1 and G_2 are defined by formulas (31) and (47), respectively.

Note that, when $\mu = 0$, the phase fluid is incompressible (div $\mathbf{v} = 0$) and its flow is steady and solenoidal (rot $\mathbf{v} \neq 0$). The direction of the phase fluid flow along the streamlines is determined by the sign of the speed v_δ.

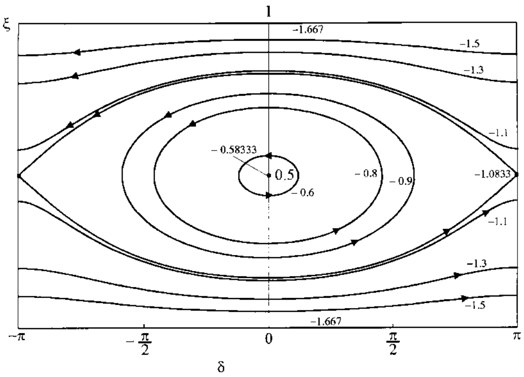

Figure 2. Phase portrait in the case of the one-to-one internal resonance $\omega_0 = \Omega_0 = 0{\cdot}3583$.

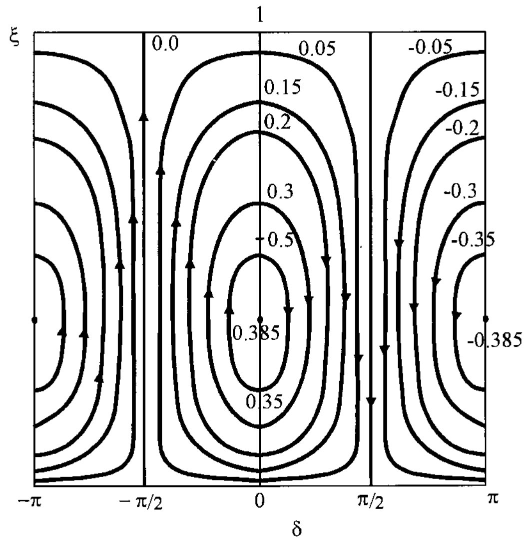

Figure 3. Phase portrait in the case of the two-to-one internal resonance $\omega_0 = 2\Omega_0$.

Plotting the stream functions in the coordinates ξ, δ for different initial conditions at the chosen magnitudes of the system's parameters, we can obtain the phase portraits shown in Figures 2 and 3 for the cases of the one-to-one and two-to-one internal resonances, respectively.

Note that similar phase portraits were obtained during the analysis of nonlinear free undamped vibrations of the Golden Gate Suspension Bridge in San Francisco for a two-to-one and one-to-one internal resonance [12].

4.1.1. *The one-to-one internal resonance*

Figure 2 shows the streamlines of the phase fluid in the phase plane for the case of the one-to-one internal resonance. Magnitudes of the value G_1 that correspond to the streamlines are indicated by digits near the curves; the flow direction of the phase fluid elements are shown by arrows on the streamlines. The phase fluid flows in an infinitely long channel, whose boundaries are the straight lines $\xi = 0$ and $\xi = 1$, corresponding to the phase modulated motions. In one part the streamlines are nonclosed, which corresponds to the periodic change of amplitudes and the aperiodic change of phases; in another part they are closed, which corresponds to the periodic change of both amplitudes and phases. The alignment of the circulation zones resembles that of Von Karman vortex streets with a symmetric arrangement. The adjacent circulation zones osculate at the saddle points with the coordinates $\xi_0^- = 0{\cdot}5$, $\delta_0 = \pi \pm 2\pi n$ $(n = 0, 1, 2, \ldots)$, $G_1 = -1{\cdot}0833$ defined by formula (33) corresponding to the unstable stationary regime. On the boundary lines of these zones (separatrixes) the value $G_1 = -13/12$, and the analytical solution corresponding to the solitonlike regime has the form

$$\log\left\|\left|\frac{2\sqrt{-0{\cdot}1154(\xi - \xi_0^-)^2 + 0{\cdot}3397} + 0{\cdot}23077}{\xi - \xi_0^-}\right|\right\|_{\xi_0}^{\xi} = \pm 0{\cdot}083 E_0 T_2, \tag{56}$$

where the sign '+' fits to the initial magnitudes $\xi_0^- < \xi_0 \leqslant 0{\cdot}8397$, $-\pi \pm 2\pi n < \delta_0 \leqslant 2\pi n$ and $0{\cdot}16032 \leqslant \xi_0 < \xi_0^-$, $-2\pi \pm 2\pi n < \delta_0 \leqslant -\pi \pm 2\pi n$, but the sign '-' conforms to the initial magnitudes $\xi_0^- < \xi_0 \leqslant 0{\cdot}8397$, $-2\pi \pm 2\pi n < \delta_0 \leqslant -\pi \pm 2\pi n$ and $0{\cdot}16032 \leqslant \xi_0 < \xi_0^-$, $-\pi \pm 2\pi n < \delta_0 \leqslant \pm 2\pi n$.

The upper branch of the separatrix describes the partial irreversible energy transfer from the vertical vibrations to the pendulum vibrations, but the lower branch, on the contrary, is in compliance with partial irreversible transfer of the energy of the pendulum vibrations to the energy of the vertical vibrations.

The motion of the fluid elements along nonclosed streamlines occurs for decreasing values of δ, but along closed streamlines in a counterclockwise direction.

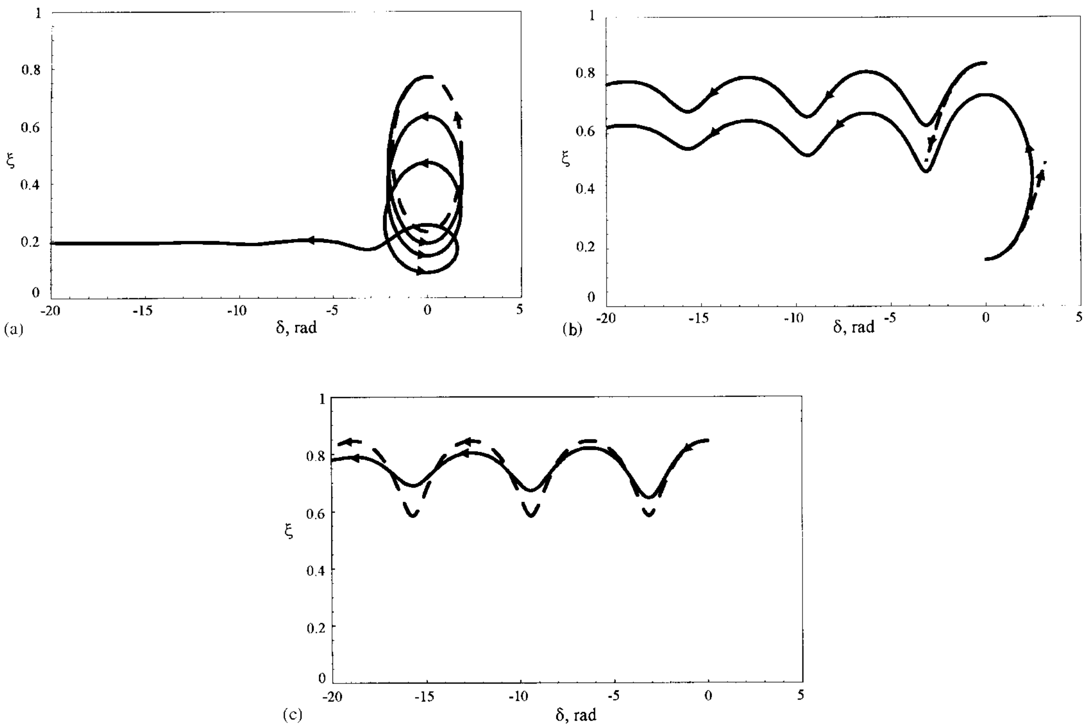

Figure 4. Trajectories of the phase fluid elements flow when $\omega_0 = \Omega_0 = 0{\cdot}3583$: (a) $\xi_0 = 0{\cdot}7703$, $\delta_0 = 0$, $G_1^0 = -0{\cdot}9$; (b) $\xi_{01} = 0{\cdot}8397$, $\xi_{02} = 0{\cdot}1603$, $\delta_{01,2} = 0$, $G_1^0 = -1{\cdot}0833$ (1 and 2 indicate the upper and lower branches of the aperiodic regime (59), respectively); (c) $\xi_0 = 0{\cdot}8453$, $\delta_0 = 0$, $G_1^0 = -1{\cdot}1$; – – – undamped and damped vibrations when $\gamma_1 = \gamma_2 = \gamma = 0{\cdot}5$; and ——— damped vibrations when $\gamma_1 = 0{\cdot}1$ and $\gamma_2 = 0{\cdot}9$.

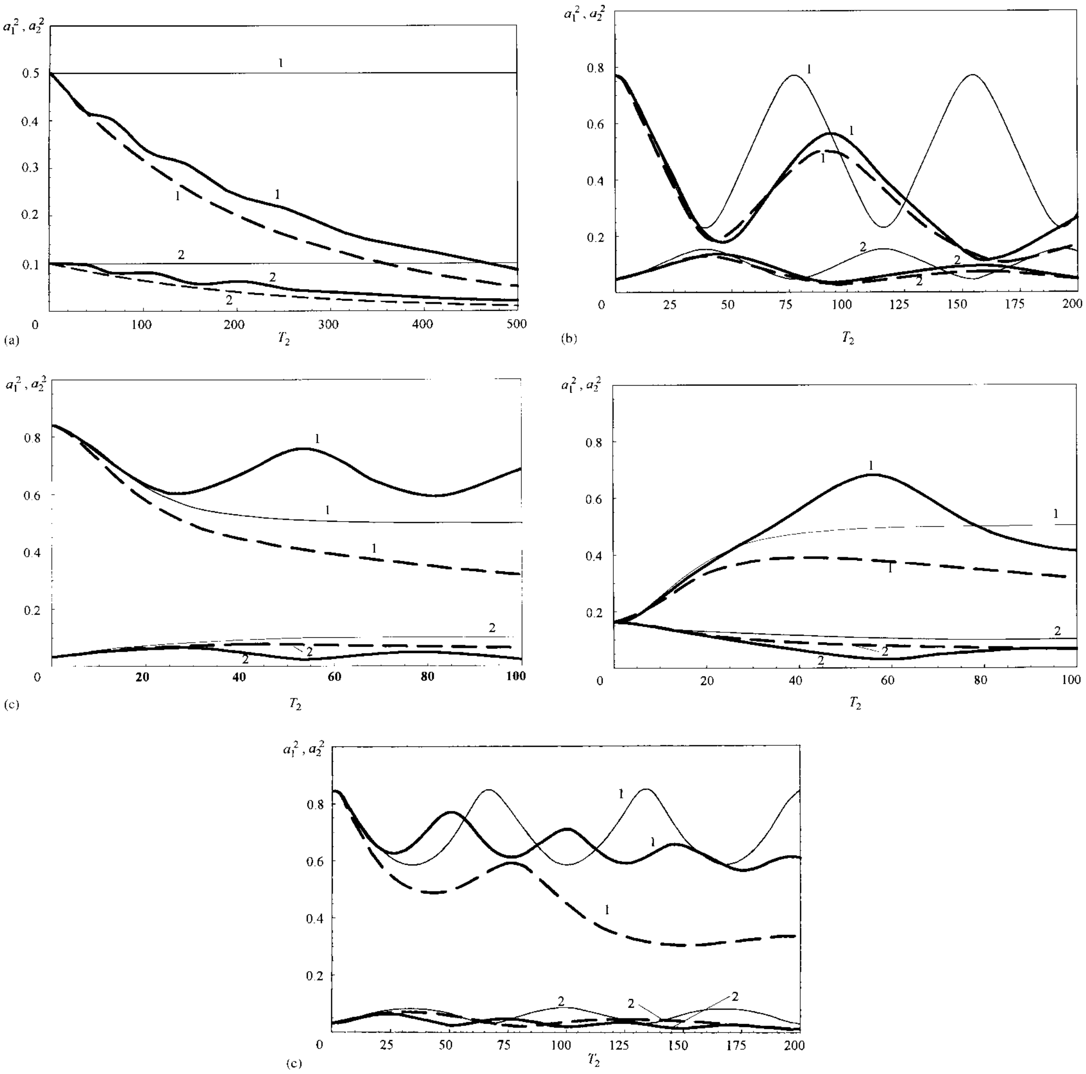

Figure 5. The time T_2 dependence of the values a_1^2 and a_2^2 at $\omega_0 = \Omega_0 = 0{\cdot}3583$: (a) $\xi_0 = 0{\cdot}5$, $\delta_0 = 0$, $G_1^0 = -0{\cdot}5833$; (b) $\xi_0 = 0{\cdot}7703$, $\delta_0 = 0$, $G_1^0 = -0{\cdot}9$; (c) $\xi_0 = 0{\cdot}8397$, $\delta_0 = 0$, $G_1^0 = -1{\cdot}0833$; (d) $\xi_0 = 0{\cdot}1603$, $\delta_0 = 0$, $G_1^0 = -1{\cdot}0833$; (e) $\xi_0 = 0{\cdot}8453$, $\delta_0 = 0$, $G_1^0 = -1{\cdot}1$; ——— undamped vibrations; – – – damped vibrations when $\gamma_1 = \gamma_2 = \gamma = 0{\cdot}5$; and ▬▬ damped vibrations when $\gamma_1 = 0{\cdot}1$ and $\gamma_2 = 0{\cdot}9$.

The points with the coordinates $\xi_0^+ = 0{\cdot}5$, $\delta_0 = \pm 2\pi n$ $(n = 0, 1, 2, \ldots)$, $G_1 = -0{\cdot}5833$ (points like a center) defined by formula (33) corresponding to the stable stationary regime are located inside the closed streamlines.

4.1.2. *The two-to-one internal resonance*

Figure 3 shows the streamlines of the phase fluid in the phase plane for the case of the two-to-one internal resonance. The boundary lines of the circulation zones tend to be located around the perimeter of the rectangle bounded by the lines $\xi = 0$, $\xi = 1$, and $\delta = \pm(\pi/2) \pm 2\pi n (n = 0, 1, 2, \ldots)$. Then the flow in each rectangle becomes isolated. On all four rectangle sides,

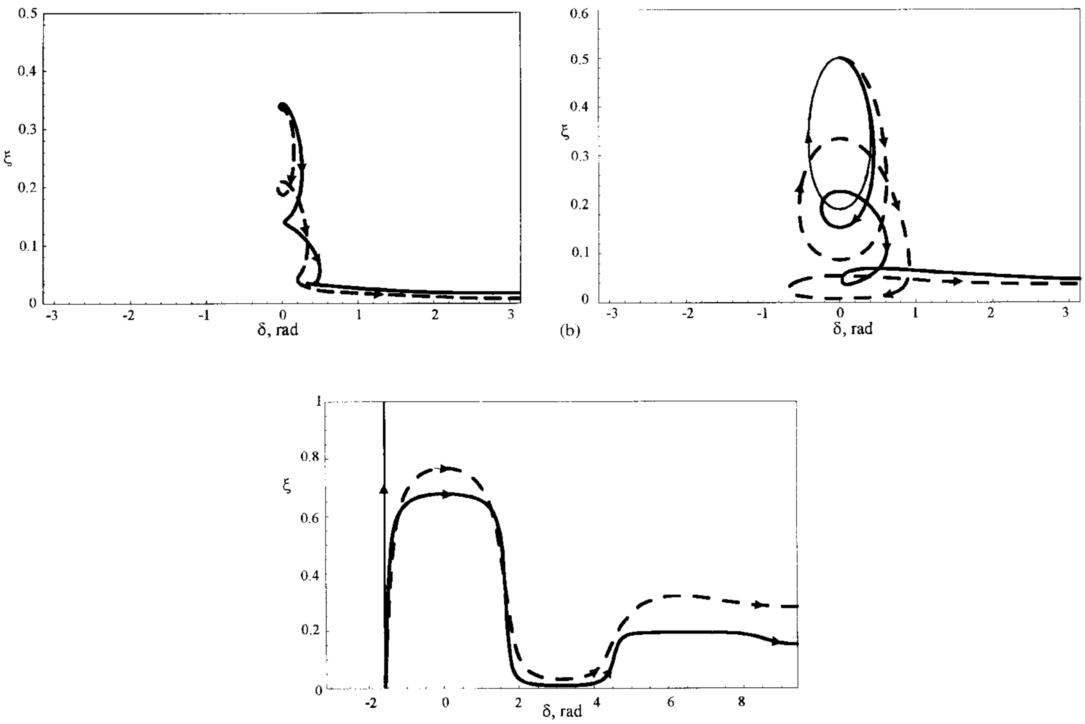

Figure 6. Trajectories of the phase fluid elements flow when $\omega_0 = 2\Omega_0 = 0{\cdot}8165$: (a) $\xi_0 = 1/3, \delta_0 = 0, G_2^0 = 0{\cdot}385$; (b) $\xi_0 = 0{\cdot}5, \delta_0 = 0, G_2^0 = 0{\cdot}3536$; (c) $\xi_0 = 0, \delta_0 = \pi/2, G_2^0 = 0$; ——— undamped vibrations; – – – damped vibrations when $\gamma_1 = \gamma_2 = \gamma = 0{\cdot}5$; and ▬▬ damped vibrations when $\gamma_1 = 0{\cdot}1$ and $\gamma_2 = 0{\cdot}9$.

$G_2 = 0$ and inside it the value of G_2 preserves its sign. Along the lines $\delta = (\pi/2) \pm \pi n$ pure amplitude modulated aperiodic motions (51) are realized; on the line $\xi = 1$ there exists the boundary phase-modulated regime. The transition of fluid elements from the points with the coordinates $\xi = 0, \delta = (\pi/2) \pm 2\pi n$ to the points $\xi = 0, \delta = -(\pi/2) \pm 2\pi n$ proceeds instantly. The points with coordinates $\xi_0^{\pm} = \frac{1}{3}, \delta_0^{\pm} = \pm 2\pi n$ defined by (53) correspond to the stable stationary regimes.

4.2. Free damped vibrations

Now let us investigate the influence of viscosity on nonlinear free vibrations of the system under consideration.

4.2.1. *The one-to-one internal resonance*

First we consider the case when $\gamma_1 = \gamma_2 = \gamma$. Then from (29) it follows that

$$\dot{\xi} = -\omega_0 b^2 E_0 \frac{\partial G_1}{\partial \delta} \exp(-sT_2), \qquad \dot{\delta} = \omega_0 b^2 E_0 \frac{\partial G_1}{\partial \xi} \exp(-sT_2). \tag{57a,b}$$

If we write the equation of a streamline

$$\frac{d\xi}{v_\xi} = \frac{d\delta}{v_\delta} \tag{58}$$

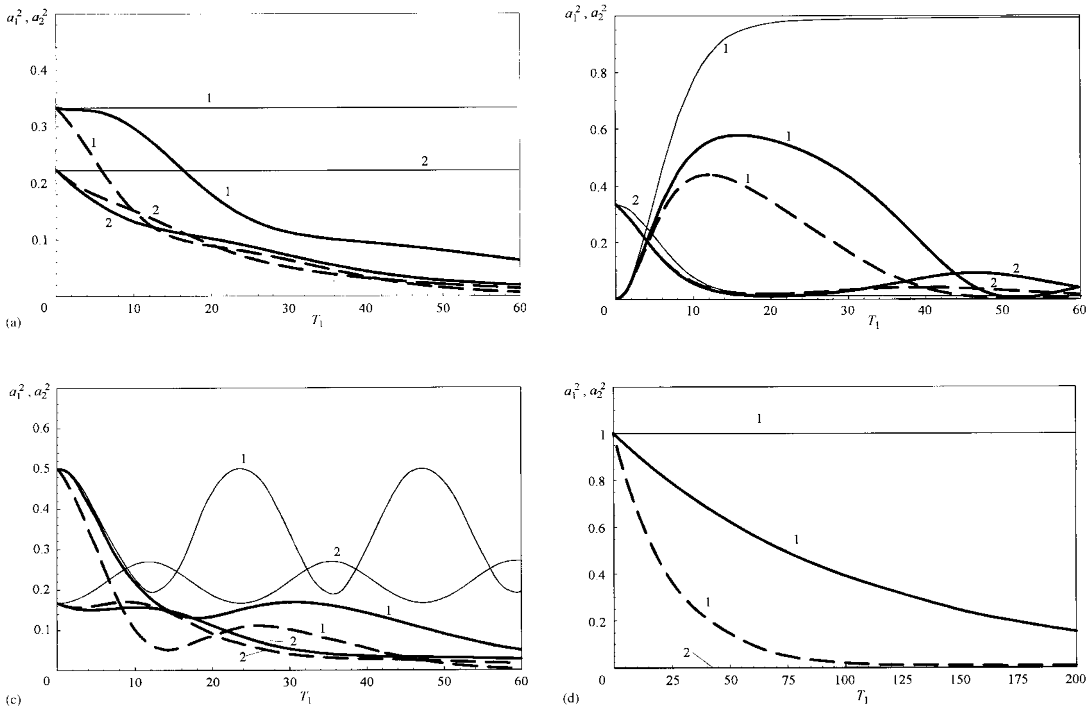

Figure 7. The time T_1 dependence of the values a_1^2 and a_2^2 at $\omega_0 = 2\Omega_0 = 0{\cdot}8165$: (a) $\xi_0 = 1/3, \delta_0 = 0, G_2^0 = 0{\cdot}385$; (b) $\xi_0 = 0, \delta_0 = \pi/2, G_2^0 = 0$; (c) $\xi_0 = 0{\cdot}5, \delta_0 = 0, G_2^0 = 0{\cdot}3536$; (d) $\xi_0 = 1, \delta_0 = 0, G_2^0 = 0$; ——— undamped vibrations; – – – damped vibrations when $\gamma_1 = \gamma_2 = \gamma = 0{\cdot}5$; and **———** damped vibrations when $\gamma_1 = 0{\cdot}1$ and $\gamma_2 = 0{\cdot}9$.

and substitute the expressions (57) in it, then we arrive at the relationship (31), which is fulfilled along each streamline. In other words, the picture of the streamlines (Figure 2) is unchanged with time, but the field of the velocities constructed along the streamlines is time dependent in such a manner that at each point $\xi,\ \delta$ of this field the direction of the velocity vector **v** remains constant, and its modulus decreases by the exponential law. In other words, in the case of the internal resonance one-to-one at $\gamma_1 = \gamma_2 = \gamma$, the flow of the phase fluid is quasi steady-stable. When this takes place, the system can execute the following three types of motion:

(1) The aperiodic damped regime without energy exchange, which corresponds to the points $\xi_0 = 0{\cdot}5,\ \delta_0 = 0,\ G_1^0 = -0{\cdot}5833$ and $\xi_0 = 0{\cdot}5,\ \delta_0 = \pm\pi,\ G_1^0 = -1{\cdot}0833$, whose coordinates are defined by the formula (28).

(2) The damped vibrational regime accompanied by two-sided energy exchange (closed and nonclosed streamlines).

(3) The aperiodic damped regime accompanied by one-sided energy exchange, which corresponds to the separatrixes with the value $G_1 = -1{\cdot}0833$, and the analytical solution for this damped soliton-like regime has the form

$$\log\left|\left|\frac{2\sqrt{-0{\cdot}1154(\xi-\xi_0^-)^2+0{\cdot}3397}+0{\cdot}23077}{\xi-\xi_0^-}\right|\right|_{\xi_0}^{\xi}$$

$$=\pm 0{\cdot}083\frac{E_0}{s}[1-\exp(-sT_2)], \tag{59}$$

where the sign '+' fits to the initial magnitudes $\xi_0^- < \xi_0 \leqslant 0{\cdot}8397$, $-\pi \pm 2\pi n < \delta_0 \leqslant 2\pi n$ and $0{\cdot}16032 \leqslant \xi_0 < \xi_0^-$, $-2\pi \pm 2\pi n < \delta_0 \leqslant -\pi \pm 2\pi n$, but the sign '−' conforms to the initial magnitudes $\xi_0^- < \xi_0 \leqslant 0{\cdot}8397$, $-2\pi \pm 2\pi n < \delta_0 \leqslant -\pi \pm 2\pi n$ and $0{\cdot}16032 \leqslant \xi_0 < \xi_0^-$, $-\pi \pm 2\pi n < \delta_0 \leqslant \pm 2\pi n$.

In this case viscosity has a stabilizing effect on the system owing to the fact that the energy exchange is quenched as time goes on.

If $\gamma_1 \neq \gamma_2$, then the streamlines disappear, i.e., the phase fluid flow becomes unsteady, and phase fluid elements during their motions begin to describe intricate trajectories, some of which are presented in Figure 4. Figures 4(a–c) present the trajectories of the phase fluid element motion at $\gamma_1 = 0{\cdot}1$ and $\gamma_2 = 0{\cdot}9$, which moves, respectively, along the closed streamline with $G_1^0 = -0{\cdot}9$, the upper and lower branches of the separatrix (59) with $G_1^0 = -1{\cdot}0833$, and along the nonclosed streamline with $G_1^0 = -1{\cdot}1$ (Figure 2) if $\mu = 0$ or $\gamma_1 = \gamma_2 = \gamma = 0{\cdot}5$. Reference to Figures 4(a–c) shows that, if $\gamma_1 \neq \gamma_2$, viscosity has a twofold effect on the system: a destabilizing influence producing unsteady energy exchange, and a stabilizing influence resulting in damping of the energy exchange mechanism.

Figure 5 shows the T_2-dependence of the square of the amplitudes a_1 and a_2 for various magnitudes of the value ξ_0 in the cases of free undamped and damped vibrations.

4.2.2. *The two-to-one internal resonance*

In the case of two-to-one internal resonance, if we introduce damping features of the system in terms of fractional derivatives, then, without regard to the magnitudes of the fractional parameters γ_1 and γ_2, the phase fluid flow becomes unsteady, and phase fluid elements during their motions begin to describe intricate trajectories, some of which are presented in Figure 6. Figures 6(a–c) present the trajectories of the phase fluid element motion at $\gamma_1 = 0{\cdot}1$ and $\gamma_2 = 0{\cdot}9$ (solid lines) and $\gamma_1 = \gamma_2 = 0{\cdot}5$ (dashed lines). These plots, in the case of undamped vibrations ($\mu = 0$), corresponds, respectively, to the stable stationary regime (53) with $G_2^0 = 0{\cdot}385$, the periodic energy exchange regime (the closed streamline with $G_2^0 = 0{\cdot}3536$), and to the aperiodic motion along the streamline $\delta = -\pi/2$ with $G_2^0 = 0$ (Figure 3). Reference to Figures 6(a–c) shows that, in the case of the two-to-one internal resonance, viscosity has a twofold effect on the system once again: a destabilizing influence producing unsteady energy exchange, and a stabilizing influence resulting in damping of the energy exchange mechanism.

Figure 7 shows the T_1-dependence of the square of the amplitudes a_1 and a_2 for various magnitudes of the value ξ_0 in the cases of free undamped and damped vibrations.

5. Conclusions

Based on the analysis carried out, the following conclusions can be deduced. The given nonlinear two-degree-of-freedom system under the conditions of the one-to-one and two-to-one internal resonances behaves differently, depending on whether it possesses damping features defined by fractional derivatives or not. Under both resonances in the absence of damping, three types of the energy exchange mechanism are observed: two-sided energy exchange (a periodic motion), one-sided energy interchange (an aperiodic motion), and energy exchange does not occur (stationary vibrations). In the phase plane free undamped vibrations of such a system correspond to steady-state motion of the phase fluid. If fractional derivatives with two independent fractional powers are introduced into the equations of motion, then during both internal resonances only periodic energy exchange accompanied by energy dissipation is observed at any initial conditions. In the phase plane such damped vibrations of the system correspond to unsteady motion of the phase fluid. Under the one-to-one internal resonance, there exists one more possibility for oscillatory motions which is realized when the fractional powers of two fractional derivatives of equal magnitude. In this case, both periodic and aperiodic energy exchange mechanisms accompanied by energy dissipation are observed. When this takes place, the aperiodic energy exchange occurs at those initial conditions which correspond to stationary and aperiodic regimes in the absence of damping. In the phase plane such damped vibratory motions correspond to the quasi-steady state motion of the phase fluid, during which the trajectories of the fluid particle motions coincide with the stream-lines; however, the velocity of the particles' motion along the stream-lines varies with time. Thus, in the case of the internal resonance, viscosity may have a twofold effect on the system: a destabilizing influence producing unsteady energy exchange, and a stabilizing influence resulting in damping of the energy exchange mechanism.

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